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Note

The number of permutations with a given signature, and the expectations of their elements

George G. Szpiro *

The Israeli Center for Academic Studies, Kiriat Ono, Israel [1]

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Abstract

This paper presents a new derivation of an enumeration formula for permutations of a given signature. Unlike the previous papers I use random number sequences which mimic the permutation, in the sense that they rise and fall as determined by the permutation's signature. Explicit expressions, previously unknown, are also derived for the mean values of the permutation's elements. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider a permutation $\{\pi_1, \pi_2, \pi_3, \dots, \pi_n\}$ of the numbers $\{1, 2, 3, \dots, n\}$. The *signature* of such a permutation is the sequence $Q = \{\kappa_1, \kappa_2, \dots, \kappa_{n-1}\}$, whose elements κ_i are equal to $+1$ or to 0, depending on whether $(\pi_{i+1} - \pi_i)$ is positive or negative. (Usually, the elements of a signature are defined to be $+1$ and -1, but for reasons that will become clear presently I shall use zero instead of -1.) It is a classical combinatorial problem to generate, or at least to enumerate the permutations of a given signature, and the results are not straightforward. The number of 'zigzag'-permutations was determined by André [1] and is still being investigated a century later, for example, by Voloshin [22] and by Randrianarivony and Zeng [18]. For general signatures Niven [16] rediscovered a determinantal solution which was originally found by MacMahon [14], and de Bruijn [7] uses this result to prove that the zigzag is the most

* Correspondence address: P.O. Box 6278, 91060 Jerusalem, Israel. Fax: ++972-2-623-02-01.
E-mail address: george@netvision.net.il (G.G. Szpiro).

[1] Affiliated with the University of Manchester.

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probable of all signatures. Carlitz [4,5] and Carlitz and Scoville [6] use Euler numbers, and Foulkes [9] uses numbers derived from the representation theory of symmetric groups to express the number of permutations with a given signature. Viennot [20,21] derives an enumeration formula based on an algorithm (which Atkinson [2] rediscovered a few years later), and Guenoche [10] describes another algorithm in terms of graph theory and 'standard tables'. Viennot's method is generalized by Kreweras and Moszkowski [13]. (See also Kreweras [12] and Etienne [8].) Mallows and Shepp [15] compute the probability that two random permutations have the same signature. Examples of papers which deal with generating algorithms are Panayatopoulos [17] (permutations with two patterns), Bauslaugh and Ruskey [3] (alternating permutations), Roelants van Baronaigien and Ruskey [19] (permutations for any given signature), and Korsh and Lipschutz [11] (permutations with repetitions).

One novelty of the present paper lies in the method of derivation. In order to obtain an explicit expression for the number of permutations of a given signature I use random number sequences which mimic the permutation, in the sense that they rise and fall as determined by the permutation's signature. As a consequence, elementary facts about the distribution of random numbers are all that is needed to compute the probabilities of such sequences. More importantly, this method also allows the computation of the expected values of the permutation's elements. To my knowledge, explicit expressions for these values have not been known previously.

2. The distribution of elements in random numbers sequences

Consider a sequence of n independent random numbers $\{Z_1, Z_2, Z_3, \ldots, Z_n\}$, where Z_i are uniformly distributed in $[0, 1]$. We define the sequence's signature in the same manner as we defined the signature of a permutation: if $Z_i > Z_{i+1}$ and $Z_{i+1} < Z_{i+2}$, we have a 'down-up' sequence whose signature is $\{0, +1\}$.

Let us analyze a sequence $\{Z_1, Z_2\}$. The first number, Z_1, can lie anywhere between zero and one. For an up-movement, the second number, Z_2, must lie between Z_1 and one; for a down-movement, it must lie between 0 and Z_1. Therefore, the probability of an upward moving 2-sequence is given by

$$P_2(U) = \int_0^1 \int_{Z_1}^1 \mathrm{d}Z_2 \, \mathrm{d}Z_1, \tag{1}$$

while for a down-movement the probability is

$$P_2(D) = \int_0^1 \int_0^{Z_1} \mathrm{d}Z_2 \, \mathrm{d}Z_1. \tag{2}$$

This can be extended to sequences of any length. Hence except for the sign,[2] the probability of an n-sequence of a certain signature, T, is given by

$$P_n(T)=\int_0^1\int_{\kappa_1}^{Z_1}\cdots\int_{\kappa_{n-1}}^{Z_{n-1}}\mathrm{d}Z_n\cdots\mathrm{d}Z_2\,\mathrm{d}Z_1, \tag{3}$$

where the integration limits, κ_i, are zero or one, depending on whether at that point the sequence moves downwards or upwards. Hence, the integration limits are precisely the elements of the sequence's signature.

Let us now consider the expected values of the random numbers for a sequence of a certain signature, i.e., the n-tuple $\{x_1,x_2,x_3,\dots,x_n\}$, where $x_i=E(Z_i)$. Since x_λ is the expected value of Z_λ, normalized by the probability that the sequence T actually occurs, its value can be computed as

$$x_\lambda=\frac{S_{n,\lambda}(T)}{P_n(T)},\quad \lambda=n,\dots,1, \tag{4}$$

where,

$$S_{n,\lambda}(T)=\int_0^1\int_{\kappa_1}^{Z_1}\cdots\int_{\kappa_{n-1}}^{Z_{n-1}}Z_\lambda\,\mathrm{d}Z_n\cdots\mathrm{d}Z_2\,\mathrm{d}Z_1,\quad \lambda=n,\dots,1. \tag{5}$$

Integrals (3) and (5) can be computed in a straightforward, although tedious, manner. (See the Appendix for an outline.) Since the κ_i only take on values of one or zero, their powers can be ignored, and we obtain,

$$\begin{aligned}P_n(T)=&+\frac{1}{n!}-\sum_{i=1}^{n-1}\frac{\kappa_i}{i!(n-i)!}+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\frac{\kappa_i\kappa_j}{i!(j-i)!(n-j)!}\\&-\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1}\frac{\kappa_i\kappa_j\kappa_k}{i!(j-i)!\,(k-j)!(n-k)!}+\cdots\end{aligned} \tag{6}$$

and

$$\begin{aligned}S_{n,\lambda}(T)=&+\frac{1}{n!}\,\frac{n+1-\lambda}{n+1}-\sum_{i=1}^{n-1}\frac{\kappa_i}{i!(n-i)!}A_{\lambda,i}+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\frac{\kappa_i\kappa_j}{i!(j-i)!(n-j)!}B_{\lambda,ij}\\&-\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1}\frac{\kappa_i\kappa_j\kappa_k}{i!(j-i)!(k-j)!(n-k)!}C_{\lambda,ijk}+\cdots,\end{aligned} \tag{7}$$

where

$$A_{\lambda,i}=\begin{cases}\dfrac{i+1-\lambda}{i+1} & \text{if } 1\leqslant\lambda\leqslant i,\\[2mm] \dfrac{n+1-\lambda}{n+1-i} & \text{if } i<\lambda\leqslant n,\end{cases}\qquad B_{\lambda,ij}=\begin{cases}\dfrac{i+1-\lambda}{i+1} & \text{if } 1\leqslant\lambda\leqslant i,\\[2mm] \dfrac{j+1-\lambda}{j+1-i} & \text{if } i<\lambda\leqslant j,\\[2mm] \dfrac{k+1-\lambda}{n+1-j} & \text{if } j<\lambda\leqslant n,\end{cases}$$

[2] When $\kappa_i=+1$ I interchange the integration limits, so that κ_I is always the lower limit. If the signature contains an odd number of $+1$'s, this produces a negative result. Since the sign plays no role in the probabilities however, we may henceforth ignore it.

$$C_{\lambda,ijk} = \begin{cases} \dfrac{i+1-\lambda}{i+1} & \text{if } 1 \leqslant \lambda \leqslant i, \\ \dfrac{j+1-\lambda}{j+1-i} & \text{if } i < \lambda \leqslant j, \\ \dfrac{k+1-\lambda}{k+1-j} & \text{if } j < \lambda \leqslant k, \\ \dfrac{n+1-\lambda}{n+1-k} & \text{if } k < \lambda \leqslant n, \ldots, \end{cases}$$

We now have expressions which permit the computation of any sequence's probability and the expected values of its elements, simply by inserting the signature's elements, κ_i, at the appropriate locations. Note that in Eqs. (6) and (7) only those terms are relevant where all κ_i are nonzero.

3. Connection to permutations

I now show that the results obtained in the previous section can be utilized to compute the number of permutations with a given signature, and also the expected values of these permutations' elements. Each sequence of n independent random numbers $\{Z_1, Z_2, Z_3, \ldots, Z_n\}$ — where Z_i are uniformly distributed in $[0,1]$ — that satisfies a certain 'up–down'-pattern, corresponds to permutations of $\{1, 2, \ldots, n\}$ with a signature which fits that pattern. Hence each sequence corresponds to a set of permutations. For example, for $n = 4$, sequences with the pattern DUD correspond to the set of permutations with signature $\{0, +1, 0\}$, i.e., the five permutations (2143, 3142, 4132, 3241, 4231).

Since random number-sequences with a certain pattern correspond to a set of permutations with a given signature, the probability of such a sequence is equal to the average of the probabilities of permutations with the corresponding signature. Each of the $n!$ permutations of $\{1, 2, 3, \ldots, n\}$ is equally likely and the same holds, of course, for permutations with a given signature. Since $P_n(T)$ of Eq. (6) is the probability of a random number-sequence with the pattern T, the number of permutations with the signature T is $(n!)P_n(T)$.[3]

The expected values of the permutation's elements can be deduced from the expectations of the elements in the random number-sequence. From the properties of the *beta*-distribution it follows that the kth-order statistic of n random variables is equal to $k/(n+1)$. Hence, in order to obtain the expectations of the permutation's elements, the expectations of the random number-sequence must be multiplied by $n+1$. For example, since the expectation of the second element in a random number-sequence of pattern DUD can be computed, by Eq. (7), as $E(Z_2) = 0.28$, the expected value of the second element in the permutations with signature $\{0, +1, 0\}$ is $5 \times 0.28 = 1.4$.

[3] See, however, Footnote 1.

4. A numerical example

To illustrate the above findings we consider, as an example, permutations of length $n=6$ with the signature $\{0,0,+1,0,+1\}$, or D^{2}UDU for short. In this case, $\kappa_1=\kappa_2=\kappa_4=0$, and $\kappa_3=\kappa_5=1$, and by Eq. (6) the probability that such a sequence occurs is

$$P_6(D^2UDU)=+\frac{1}{6!}-\frac{1}{3!3!}-\frac{1}{5!1!}+\frac{1}{3!2!1!}=\frac{7}{144}.$$

Since the total number of permutations is $n!$, the number of permutations of signature D^{2}UDU is 35.

The mean value of the elements of the D^{2}UDU-permutation can be computed by Eqs. (6) and (7) in the following manner:

$$S_{6,1}(D^2UDU)=+\frac{6}{6!7}-\frac{3}{3!3!4}-\frac{5}{5!1!6}+\frac{3}{3!2!1!4}=\frac{181}{5040},$$

$$S_{6,2}(D^2UDU)=+\frac{5}{6!7}-\frac{2}{3!3!4}-\frac{4}{5!1!6}+\frac{2}{3!2!1!4}=\frac{117}{5040},$$

$$S_{6,3}(D^2UDU)=+\frac{4}{6!7}-\frac{1}{3!3!4}-\frac{3}{5!1!6}+\frac{1}{3!2!1!4}=\frac{53}{5040},$$

$$S_{6,4}(D^2UDU)=+\frac{3}{6!7}-\frac{3}{3!3!4}-\frac{2}{5!1!6}+\frac{2}{3!2!1!3}=\frac{164}{5040},$$

$$S_{6,5}(D^2UDU)=+\frac{2}{6!7}-\frac{2}{3!3!4}-\frac{1}{5!1!6}+\frac{1}{3!2!1!3}=\frac{65}{5040},$$

$$S_{6,6}(D^2UDU)=+\frac{1}{6!7}-\frac{1}{3!3!4}-\frac{1}{5!1!2}+\frac{1}{3!2!1!2}=\frac{155}{5040}.$$

After dividing these numbers by $7/144$ ($=P_6(D^2UDU)$), and normalizing (by multiplying the results with 7), we obtain the mean values $x_1=181/35=5.17$, $x_2=117/35=3.34$, $x_3=53/35=1.51$, $x_4=164/35=4.69$, $x_5=65/35=1.86$, and $x_6=155/35=4.43$.

Let us now compare these results with the actual situation. The following table lists the 35 permutations of $\{1,2,\ldots,6\}$ which have the signature D^2UDU:

6 5 3 4 1 2	6 3 2 5 1 4	6 3 1 5 2 4	3 2 1 6 4 5	4 3 2 5 1 6	4 3 1 5 2 6
6 4 3 5 1 2	5 3 2 6 1 4	5 3 1 6 2 4	6 5 2 3 1 4	6 5 1 3 2 4	6 2 1 4 3 5
5 4 3 6 1 2	4 3 2 6 1 5	4 3 1 6 2 5	6 4 2 3 1 5	6 4 1 3 2 5	5 2 1 4 3 6
6 5 2 4 1 3	6 5 1 4 2 3	6 2 1 5 3 4	5 4 2 3 1 6	5 4 1 3 2 6	4 2 1 5 3 6
6 4 2 5 1 3	6 4 1 5 2 3	5 2 1 6 3 4	6 3 2 4 1 5	6 3 1 4 2 5	3 2 1 5 4 6
5 4 2 6 1 3	5 4 1 6 2 3	4 2 1 6 3 5	5 3 2 4 1 6	5 3 1 4 2 6	

The mean values of the elements are, as expected, $x_1=5.17$, $x_2=3.34$, $x_3=1.51$, $x_4=4.69$, $x_5=1.86$, and $x_6=4.43$.

5. Conclusion

The enumeration of permutations of a given signature is a classical combinatorial problem. Various procedures have been found to compute the number of such permutations. This paper proposes a novel method, based on an analysis of random number-sequences whose up- and down-patterns mimic the permutation's signature. Explicit formulas are derived for the probability of random number-sequence with specified patterns, which are then linked to permutations with a given signature. Previously unknown formulas for the expectations of these permutations' elements are obtained as a byproduct.

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Appendix

In this appendix I show in outline how the integral in Eq. (3) is computed to obtain the result in Eq. (6). Eq. (7) can be derived from integral (5) in a similar manner. We start with Eq. (3),

$$P_n(T) = \int_0^1 \int_{\kappa_1}^{Z_1} \cdots \int_{\kappa_{n-1}}^{Z_{n-1}} \mathrm{d}Z_n \cdots \mathrm{d}Z_2\, \mathrm{d}Z_1$$

and first solve the innermost integral

$$\int_{\kappa_{n-1}}^{Z_{n-1}} \mathrm{d}Z_n = Z_n\Big|_{\kappa_{n-1}}^{Z_{n-1}} = Z_{n-1} - \kappa_{n-1}.$$

Now the next-innermost integral can be computed

$$\int_{\kappa-2}^{Z_{n-2}} (Z_{n-1} - \kappa_{n-1})\, \mathrm{d}Z_{n-1} = \left(\frac{(Z_{n-1})^2}{2} - \kappa_{n-1} Z_{n-1} \right)\Bigg|_{\kappa_{n-2}}^{Z_{n-2}}$$

$$= \frac{(Z_{n-2})^2}{2} - \frac{(\kappa_{n-2})^2}{2} - [\kappa_{n-1} Z_{n-2} - \kappa_{n-1}\kappa_{n-2}].$$

We also illustrate the third integral

$$\int_{\kappa-3}^{Z_{n-3}} \left(\frac{(Z_{n-2})^2}{2} - \frac{(\kappa_{n-2})^2}{2} - [\kappa_{n-1} Z_{n-2} - \kappa_{n-1}\kappa_{n-2}] \right) \mathrm{d}Z_{n-2}$$

$$= \frac{(Z_{n-2})^3}{3 \cdot 2} - \frac{(\kappa_{n-2})^2 Z_{n-2}}{2} - \left[\frac{\kappa_{n-1}(Z_{n-2})^2}{2} - \kappa_{n-1}\kappa_{n-2} Z_{n-2} \right]\Bigg|_{\kappa_{n-3}}^{Z_{n-3}}$$

$$
\begin{aligned}
&= \frac{(Z_{n-3})^3}{3 \cdot 2} - \frac{(\kappa_{n-3})^3}{3 \cdot 2} - \left[\frac{(\kappa_{n-2})^2 Z_{n-3}}{2} - \frac{(\kappa_{n-2})^2 \kappa_{n-3}}{2} \right] \\
&\quad - \left[\frac{\kappa_{n-1}(Z_{n-3})^2}{2} - \frac{\kappa_{n-1}(\kappa_{n-3})^2}{2} \right] + [\kappa_{n-1}\kappa_{n-2}Z_{n-3} - \kappa_{n-1}\kappa_{n-2}\kappa_{n-3}].
\end{aligned}
$$

The further development quickly becomes unwieldy. After integrating k times we have 2^k terms which are composed of powers of Z_{n-k} and powers of $\kappa_{n-\lambda}$. Since during each integration one factor is adjoined to the term, the sum of the powers equals k. The numerator is the product of the faculties of all powers contained in the term. Hence, a general term is as follows:

$$
\frac{Z_{n-k}^j}{j!} \frac{\kappa_{n-r}^p}{p!} \frac{\kappa_{n-r+1}^q}{q!} \cdots \quad \text{where } 0 \leqslant j \leqslant k, \text{ and } j + p + q + \cdots = k.
$$

More information about the structure of the terms could be given – for example $j + r$ must equal k — but that would lead us too far afield. The results simplify somewhat, once we realize that the integration limits, $\kappa_{n-\lambda}$, can only take on the values zero or one, and that powers of $\kappa_{n-\lambda}$ can, therefore, be ignored. I omit further details except to mention that this process continues until we reach the outermost integral:

$$
\begin{aligned}
&\int_0^1 \frac{(Z_1)^{n-1}}{(n-1)!} + \cdots + \kappa_{n-1}\kappa_{n-2}\ldots\kappa_1 \, \mathrm{d}Z_1 \\
&= \frac{Z_1^n}{n!} + \cdots + \kappa_{n-1}\kappa_{n-2}\ldots\kappa_1 Z_1 \Big|_0^1 \\
&= \frac{1}{n!} + \cdots + \kappa_{n-1}\kappa_{n-2}\ldots\kappa_1,
\end{aligned}
$$

which, after re-arranging, corresponds to Eq. (6).

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